Making Mathematical Connections to the Order of Operations: Supportive and Problematic Conceptions

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Mathematical connections play a critical role in developing a deeper understanding about mathematics, but misinterpretations may arise when connections are based on problematic conceptions. This paper reports on findings from a questionnaire and follow-up interviews with pre-service secondary mathematics teachers. Chin's (2013) Supportive and Problematic Conceptions framework is used to analyse the basis of pre-service teachers' conceptions when they establish mathematical connections pertaining to the order of operations. Findings reveal that the pre-service teachers connected the order conventions to the properties of operations and to the set concept. Findings also show that the conception of a quadratic equation as having two roots was problematic in that it impeded sense making of the order of operations.

The order of operations is generally seen as an arbitrary convention and taught by rote using mnemonics such as BODMAS or its variants (Zazkis & Rouleau, 2018). As a consequence, it causes misinterpretations such as division takes precedence over multiplication and raises concerns about the conceptual difficulties of the order of operations (Chin et al., 2022). To avoid these misinterpretations, there are suggestions that learning of the order of operations would benefit from a strong grasp of connections between the topic and other mathematical ideas (e.g., Coles & Sinclair, 2019; Toh & Choy, 2021). Viewing the order of operations as being connected with other mathematical ideas, such as the properties of operations could potentially develop "the total cognitive structure" (Tall & Vinner, 1981, p. 152) that is associated with the order precedence.

In a general sense, a mathematical connection is a relationship between mathematical ideas. In essence, mathematical connections or specifically known as *intra-mathematical connections* concern relationships between "representations, definitions, concepts, procedures and propositions within the context of mathematics" (Gamboa et al., 2021, p. 4). Some researchers propose *extra-mathematical connections* as referring to relationships between mathematics and contexts outside mathematics (e.g., Gamboa et al., 2021; NCTM, 2000). In this paper, the exploration of mathematical connections is in relation to intra-mathematical connections, that is, the mathematical ideas pre-service teachers connect when working with order of operations tasks.

Making mathematical connections is imperative because it plays a critical role in developing theoretical thinking and a deeper understanding about mathematics (Cai et al., 2014; Dorier & Sierpinska, 2001). The importance of making connections is also highlighted in some eminent teacher frameworks such as the Knowledge Quartet (KQ) and the Teaching for Robust Understanding (TRU) (Rowland et al., 2005; Schoenfeld, 2013). The TRU framework, for example, demands a classroom discussion to provide opportunities for making coherent connections between mathematical ideas (Schoenfeld, 2013). However, it is questionable what connections pre-service teachers make to the order of operations since it is commonly perceived as arbitrary.

Previous studies in relation to making connections in mathematics have focussed on conceptualisation of mathematical connections and the potential of using connections in support of student learning (e.g., Eli et al., 2011; García-García & Dolores-Flores, 2021; Gamboa et al., 2020, Rodríguez-Nieto et al., 2022). However, not all connections made are useful for learning. Although some mathematical connections are useful in helping students to better understand a mathematical concept, some might be problematic as the connections hinder students from making sense of the concept correctly. The current study differs from existing research in which this study analyses, not

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only the connections that are supportive, but also the connections that are problematic and impede learning.

As part of a larger research project, this paper aims to understand the connections pre-service teachers make to the order of operations. The research questions addressed in this paper are:

- What mathematical connections do pre-service teachers make to the order of operations?
- How do the mathematical connections affect the pre-service teachers' evaluations of mathematical expressions?

This study is significant for several reasons. Other than extending the literature about mathematical connections by analysing those that can support or impede learning, this study also documents intra-mathematical connections that can be made to the order of operations.

Theoretical Perspectives

Sfard (1991) described a *conception* as "the whole cluster of internal representations and associations evoked by the concept—the concept's counterpart in the internal, subjective "universe of human knowing" (p. 3). This implies that a conception is one of the building blocks that contributes to the divergence of performance in doing mathematics (Chin, 2013). In considering the conceptions in mathematics learning, Chin (2013) proposes supportive and problematic conceptions to describe how the personal conceptions affect the sense making of mathematical concepts. A supportive conception is a conception that supports generalisation whereas a problematic conception is a conception that impedes sense making.

Research examining conceptions of learning has dominated with studies about how previous knowledge impacts new learning (e.g., Bransford & Schwartz, 1999; Jiew & Chin, 2020; Wagner, 2010). Although it is agreed that much can be gained from analysing the influences of prior experience to what is learned later, such work should be complementary to research understanding how newly developed conceptions affects prior learning. In this field of research, Lima and Tall (2008) uses *met-after* to describe an experience encountered at a later learning stage that impacts the memories of old learning. When solving linear equations, some student participants in Lima and Tall's (2008) study misapplied the quadratic formula to solve the equations, and some made sense of the equals sign as multiplication. These were instances of *met-after*. As students were introduced to linear equations before quadratic equations, their experience working with the quadratic formula influenced what they remembered about linear equations. Lima and Tall (2008), however, refers to experience in general but not the conceptions developed within linear or quadratic equations.

Another construct proposed by Hohensee (2014) that aligns with *met-after* is *backward transfer*. It explains "the influence that constructing and subsequently generalizing new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (p. 136). Hohensee's (2014) construct, however, refers to ways of reasoning that students engage in doing mathematics. I suggest that the effects of new learning on old learning may also include the underlying mental structures such as conceptions of the mathematical concept itself.

In this paper, we use the notions of supportive and problematic conceptions to look closely at the basis of pre-service teachers' conceptions when they established mathematical connections to the order of operations. The framework of supportive and problematic conceptions may also provide insights into how the mathematical connections affect the evaluations of mathematical expressions. Knowing what mathematical connections are supportive is significant as it leads to meaningful learning. Knowing what conceptions are problematic is also essential as it avoids misinterpretations of a mathematical concept.

Methodology

Setting and Participants

This study is conducted as the first author's PhD research. There were 11 pre-service secondary mathematic teachers who were towards the end of their 4-year pre-service teacher education. All participants gave informed consent. Data collection involved a questionnaire and follow-up interviews. Some findings from the questionnaire and interview data are reported in this paper. Particularly, this paper discusses the participants' responses about two mathematical expressions involving multiplication and division as follows:

$$10 \div 5 \times 2$$

$$4 \times 6 \div 3$$

Data Collection

A questionnaire about order of operations tasks was administered and follow-up interviews were conducted via the Zoom platform. The online platform was used because of prevailing COVID-19 restrictions at that time, which prevented in-person meetings from taking place. Follow-up questions were asked in the interviews to gain further insights into the participants' explanation about their responses provided in the questionnaire. The research session lasted between 50 and 60 minutes. Interviews were recorded and subsequently transcribed for analysis.

Data Analysis

Data were analysed thematically following the steps suggested by Ary et al. (2013) and using the approaches proposed by Braun and Clarke (2021). The data were organised and annotated to identify responses that provided information about ways of evaluation of mathematical expressions and mathematical connections the participants made. Codes were generated to represent the most relevant data for the research questions. We used both "data-driven" and "theory-driven" approaches to avoid missing important data. After repeated iterations of coding, codes were then used to interpret themes. The generated themes were used to inform an overall understanding of the mathematical connections made by the participants.

Results

The correct order to evaluate the expressions containing multiplication and division is from left to right. Of the 11 participants, six used the correct order, three performed division before multiplication, and two evaluated the expressions without a specific order. In this respect, evaluating an expression using no specific order means providing two different solutions to the expression. In other words, the participants evaluated the expressions without giving priority to any operation. The different orders of evaluation are discussed in turn.

From Left to Right

Brendan's responses are reproduced in Figure 1 to illustrate the way that the six participants used the left-to-right order.

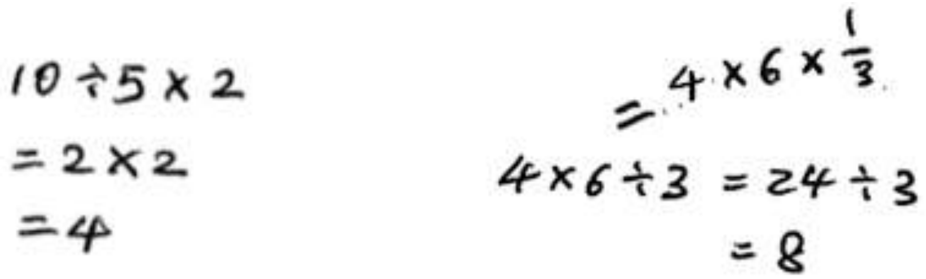

Figure 1. Evaluate the expressions from left to right—Brendan's responses.

Brendan's explained that, "We can write division as multiplication. When they are all in multiplication, we can change the place, so any order doesn't matter. But when we maintain the question in multiplication and division, we have to follow the left-to-right order." Brendan connected the order of operations to properties of operations, particularly the multiplicative inverse and the associative property. Writing $\div 3 = \times \frac{1}{3}$ for the expression $4 \times 6 \div 3$ implies that he recognised division as the inverse of multiplication. Based on the associativity of multiplication, he realised that the numbers in an expression could be shifted if division was written as multiplicative. His conceptions about multiplicative inverse and associativity of multiplication support his explanation of using the left-to-right order. These conceptions are considered as supportive conceptions that allow Brendan to make generalisations.

On the other hand, Julie based her explanation on another mathematical concept, namely that of a set. She wrote $A \cup B \cap C$ and stated that, "For mixed operations, we must do from left to right. For example, A union B and intersection C. We need to do it from the left first, which is the union first." Julie linked the order of operations to set notation and set operations. Specifically, her conception about set notations and set operations is a supportive conception that allows her to successfully interpret the order of operations. This conception supports her generalisation in the context of simplifying numerical expressions.

Division Before Multiplication

Audrey's responses are reproduced in Figure 2 to illustrate the way that the three participants used division before multiplication. For expression $4 \times 6 \div 3$, Audrey computed the right answer but used the wrong order.

Figure 2. Perform division before multiplication—Audrey's responses.

The participants who used division before multiplication explained their responses based on the acronym BODMAS. Audrey, for example, stated that, "Following BODMAS, D division has to be calculated before M multiplication. DM means division then Multiplication." Audrey connected the order of operations to another representation of the rules that is BODMAS. However, she misinterpreted that division precedes multiplication based on the order in which the letters were presented in the acronym. As the letter D comes before the letter M, Audrey perceived it as division takes precedence over multiplication. Even though the final answer is correct, she had a misinterpretation of the acronym.

No Specific Order

Howard's responses are reproduced in Figure 3 to illustrate the way that the two participants evaluated the expressions with no specific order.

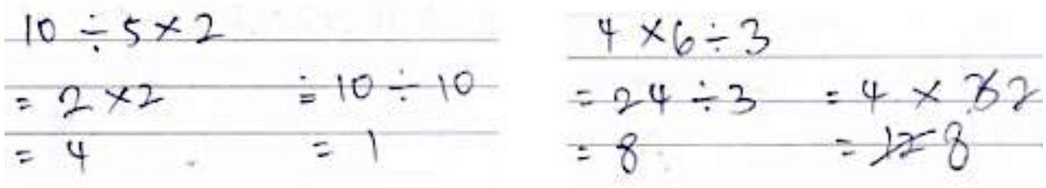

Figure 3. Evaluate the expressions with no specific order.

Howard explained that, "This kind of questions can have two answers. Just like an unknown, we can have two answers, for example x equals to 1 and -1, sometimes we can have 4 answers." He explained his responses based on his conception of algebra. Obtaining two answers from an unknown implies that Howard made sense of the order of operations based on his understanding that a quadratic equation could have two roots. His conception of quadratic equation is correct in the context of algebra, but it is problematic in the context of simplifying numerical expressions. This conception is a problematic conception that impedes his sense making of the order of operations.

Discussion and Concluding Remarks

The findings suggest that pre-service teachers made sense of the left-to-right order based on properties of operations. Their conceptions about associative and inverse properties are supportive conceptions that help them make sense of the order of operations. Building on these connections, they were able to perform calculations using the correct order. This finding contradicts Zazkis and Rouleau's (2018) argument, namely, the knowledge about the properties of operations was not used by pre-service teachers to interpret the order of operations. Another supportive conception revealed in this study is when Julie connected the order of operations to set notation and set operations. These connections work within the set context and continue to support the sense making of the order of operations.

The conception that a quadratic equation can have two roots is a problematic conception in the context of simplifying numerical expression. Although this conception is true and workable in the context of algebra, in particular with quadratic equation, this conception does not work when completing order of operations tasks in linear equations. It impedes the sense making of the order of operations.

The findings that making supportive connections to set and algebra align with Hohensee's (2014) notion of *backward transfer* and Lima and Tall's (2013) *met-after*. This shows that learning is not necessarily from simple to complex, it may be that new learning impacts on prior knowledge. Both the studies found that new learning of quadratic functions affected students' understanding of linear functions. The present study extended these prior studies in another mathematical area (i.e., the order of operations) and included conceptions rather than experience or ways of reasoning.

There was evidence that the pre-service teachers evaluated expressions based on the acronym BODMAS, which led to a misinterpretation. Although the use of an acronym in teaching the order conventions may lead to proficiency in recalling and performing the necessary calculations or procedures, the findings reveal the danger of using acronyms in the order of operations as reported in existing research (e.g., Dupree, 2016; Glidden, 2008; Zazkis & Rouleau, 2018).

Making mathematical connections is a key goal in the learning of mathematics (NCTM, 2009) and researchers encourage connection-making in learning mathematics with understanding (Bossé, 2003; Cai & Ding, 2017). In this study, further insights were gained from knowing the basis of conceptions when mathematical ideas were connected. Supportive conceptions are necessary to bring meaningful learning whereas problematic conceptions are essential to prevent misinterpretations of a mathematical concept or idea.

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